Microwave Faraday Effect and Propagation in a Circular Solid-State Plasma Waveguide

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Abstract—The microwave Faraday effect in a solid-state plasma waveguide under a relatively high magnetic field ($\mu_e B_0 \gg 1$) is investigated. Field configurations in the solid-state plasma waveguide are first analyzed. It is shown that two types of nondegenerate circularly polarized quasi-TE waves exist in the waveguide. The propagation constants of the quasi-TE waves are obtained by means of a variational method. The microwave Faraday effect in the solid-state plasma waveguide is formulated. It is shown that geometrical factors and reflections from the waveguide discontinuities have significant influences on the Faraday effect.

Experimental results of the Faraday effect at 36 GHz through a solid-state plasma waveguide are reported. In the experiment n-type indium antimonide crystals are used. The comparison of the experimental data with the theory shows good agreement.

I. Introduction

The conductivity of a solid-state plasma becomes a tensor quantity under an externally applied magnetic field. If a dc magnetic field B_0 is applied along the z axis, the tensor conductivity may be given by

$$\bar{\sigma} = \begin{bmatrix} \sigma_\perp, & -\sigma_x, & 0 \\ \sigma_x, & \sigma_\perp, & 0 \\ 0, & 0, & \sigma_\parallel \end{bmatrix} \tag{1}$$

where, if it is assumed that $n_e\mu_e \gg n_h\mu_h$ as in an n-type semiconductor, we have $\sigma_\parallel = (\alpha_e n_e \mu_e)(1+j\omega\tau_e)^{-1}$, $\sigma_\perp = (\sigma_\parallel)(1+\alpha_e^2)^{-1}$, $\sigma_x = (\sigma_\parallel \alpha_e)(1+\alpha_e^2)^{-1}$, $\alpha_e = (\mu_e B_0)(1+j\omega\tau_e)^{-1}$, and q is the magnitude of electron charge, n the density, μ the mobility, τ the relaxation time constant, ω the angular frequency, and $j=\sqrt{-1}$ the unit imaginary number. The subscripts e and h refer to electrons and holes, respectively. Fig. 1 shows variations of $\sigma_\parallel$, $\sigma_\perp$, and σ_x as a function of B_0.

In such an anisotropic unbound medium, two types of circularly polarized TEM waves, viz., left- and right-hand circularly polarized waves, propagate along the magnetic field with different propagation constants k_+ and k_- (which are complex quantities in general) given by [1]–[9]

$$k_\pm^2 = \omega^2\mu_0\epsilon - j\omega\mu_0\sigma_\perp \pm \omega\mu_0\sigma_x \tag{2}$$

where μ_0 is the permeability and ϵ is the dielectric constant of the medium.

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A linearly polarized wave can be considered to consist of the two circularly polarized waves of equal amplitude. Thus, when a linearly polarized electromagnetic wave propagating along the magnetic field enters a solid-state plasma slab, the wave will be decomposed into two such circularly polarized waves, which will then propagate with different phase velocities giving a rise to a rotation of the plane of polarization as the wave propagates. The angle of rotation of the plane of polarization θ and the ellipticity of the polarization of the transmitted wave E are, respectively, given by [1]–[8]

$$\theta = (\beta_+ - \beta_-)(\ell/2) \tag{3}$$

$$E = \tanh\left[(\alpha_- - \alpha_+)(\ell/2)\right] \tag{4}$$

where β and α are, respectively, the real and imaginary parts of the propagation constant, and ℓ is the thickness of the solid-state plasma slab.

This rotation is similar to the phenomenon known as the Faraday rotation in optics[10] which is thus called microwave Faraday rotation when it occurs in a microwave frequency region. The microwave Faraday effects in some semiconductors such as germanium and silicon have been investigated[1]–[9] primarily in the region where the Hall angle is small, i.e., $\mu_e B_0 < 1$. In this region, σ_x is small and nearly directly proportional to B_0, thus, θ is a linear function of B_0, and the quantity $V = \theta/B_0\,\ell$ is called the Verdet constant. Since σ_x is relatively small, and $\sigma_\perp$ is very large in this region, the magnitude of V is very small and the attenuation of the wave is very large. Effects of the wave reflections on the Faraday rotation were treated by Donovan and Medcalf[9] in this region.

In the region where the Hall angle is relatively large, i.e., $\mu_e B_0 \gg 1$, the propagation constants are usually approximated by [11]–[14]

$$k_\pm^2 \approx \pm\,\omega\mu_0\sigma_x. \tag{5}$$

In this approximation, it is assumed that the conductivity of the solid-state plasma is so high that the displacement current is negligibly small, i.e., $\omega\epsilon < \sigma_x$. Under this condition, one type of circularly polarized wave can propagate with very small attenuation, but the other wave is cut off. The propagating circularly polarized wave is called a helicon wave;[11] propagation of helicon waves has been investigated also in vacuum bound semiconductors.[15],[16]

For a relatively high frequency region, however, a condition can be obtained such that $\omega^2\mu_0\epsilon > \omega\mu_0\sigma_x \gg \omega\mu_0\sigma_\perp$ pro-

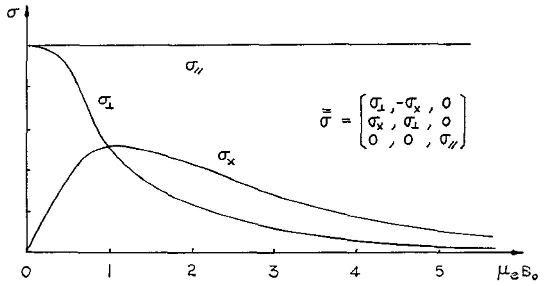

Fig. 1. Variation of the tensor conductivity of a solid-state plasma as a function of applied magnetic field.

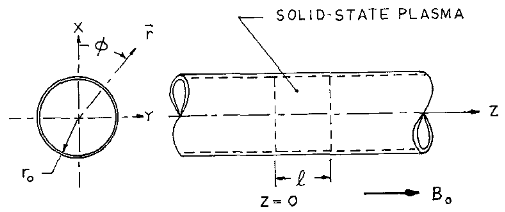

Fig. 2. A circular solid-state plasma waveguide.

vided that $\mu_e B_0 \gg 1$. Under this condition both circularly polarized waves can propagate,[17],[18] and the Faraday rotation with very small attenuation can be obtained.

In most reported analyses of microwave Faraday effect in semiconductors,[1]–[9],[18] uniform TEM waves in an unbound space were assumed. In cylindrical waveguides, however, the TEM mode cannot exist. In this paper, a theoretical analysis and experimental investigation of the microwave Faraday effect in a solid-state plasma waveguide (rather than in an infinite slab), placed under a relatively high longitudinal magnetic field as shown in Fig. 2 are presented. Field configurations in the waveguide are first analyzed and propagation constants are obtained. The Faraday effect that arises in the waveguide is then formulated and compared with the experimental results. Effects of the wave reflections on the Faraday rotation are also taken into consideration.

In the analysis the following assumptions will be made:

1) Effect of holes is negligibly small in comparison with that of electrons, i.e., $n_e\mu_e \gg n_h\mu_h$, as in n-type semiconductors.
2) The mobility μ_e is constant.
3) The applied dc magnetic field B_0 is relatively high so that the Hall angle is very large, i.e., $\mu_e B_0 \gg 1$.
4) The frequency of the microwave signal is much smaller than both the cyclotron and plasma resonant frequencies, i.e., $\omega \ll \omega_c$, $\omega \ll \omega_p$.
5) The wall of the waveguide is perfectly conductive so that the tangential component of the electric field vanishes at the wall.

The analysis can, however, be extended readily to other cases.

II. Field Configurations

An electromagnetic wave in a solid-state plasma under a dc magnetic field is governed by Maxwell's equations of the form

$$\nabla \times \boldsymbol{E} = -j\omega\mu_0 \boldsymbol{H} \tag{6a}$$

$$\nabla \times \boldsymbol{H} = j\omega\epsilon \boldsymbol{E} + \boldsymbol{i} \tag{6b}$$

$$\nabla \cdot (\epsilon \boldsymbol{E}) = \rho \tag{6c}$$

$$\nabla \cdot (\mu_0 \boldsymbol{H}) = 0 \tag{6d}$$

where the conduction current is given in terms of the tensor conductivity and the electric field, i.e.,

$$\boldsymbol{i} = \bar{\bar{\sigma}} \cdot \boldsymbol{E}. \tag{7}$$

The wave propagating the waveguide along the z axis may be expressed by the field components of the form

$$\boldsymbol{E}(r, \phi, z, t) = [\boldsymbol{E}_t(r, \phi) + \boldsymbol{E}_z(r, \phi)]e^{j(\omega t - kz)} \tag{8}$$

$$\boldsymbol{H}(r, \phi, z, t) = [\boldsymbol{H}_t(r, \phi) + \boldsymbol{H}_z(r, \phi)]e^{j(\omega t - kz)} \tag{9}$$

where k is the propagation constant and the subscripts t and z refer to the transverse and longitudinal components, respectively.

Substituting (8) and (9) into (6) and (7), and separating the transverse and longitudinal components, then eliminating the transverse components $\boldsymbol{E}_t$ and $\boldsymbol{H}_t$, we obtain[19],[20]

$$\left[\nabla_t^2 - k^2\left(\frac{j\omega\epsilon + \sigma_\parallel}{j\omega\epsilon + \sigma_\perp}\right) + (\omega^2\mu_0\epsilon - j\omega\mu_0\sigma_\parallel)\right]E_z - jk\left(\frac{j\omega\mu_0\sigma_x}{j\omega\epsilon + \sigma_\perp}\right)H_z = 0 \tag{10}$$

$$\left[\nabla_t^2 - k^2 + \omega^2\mu_0\epsilon - j\omega\mu_0\sigma_\perp - \left\{\frac{j\omega\mu_0\sigma_x^2}{j\omega\epsilon + \sigma_\perp}\right\}\right]H_z + jk\left[\sigma_x\frac{(j\omega\epsilon + \sigma_\parallel)}{(j\omega\epsilon + \sigma_x)}\right]E_z = 0 \tag{11}$$

where ∇_t represents the vector differential operator in the transverse directions.

Equations (10) and (11) must be satisfied simultaneously by the same values of the propagation constant k. There are, in general, no solutions that are of pure TE_z or TM_z modes which satisfy (10) and (11) simultaneously, unless $\sigma_x = 0$.[19],[20] The differential equation for the transverse dependence of k in general becomes fourth order.[15],[19],[20] Noting, however, that $\sigma_\parallel$ is very large in comparison with other terms in (10) and (11) for a solid-state plasma, in view of the assumption 3), (10) may be reduced to

$$(\omega^2\mu_0\epsilon - j\omega\mu_0\sigma_\perp - k^2)\sigma_\parallel E_z + k\omega\mu_0\sigma_x H_z \approx 0. \tag{12}$$

Eliminating $\boldsymbol{E}_z$ from (11) and (12), we get

$$\nabla_t^2 H_z + k_T^2 H_z = 0 \tag{13}$$

where we let

$$k_T^2 = (\omega^2\mu_0\epsilon - j\omega\mu_0\sigma_\perp - k^2) - \left\{\frac{(\omega\mu_0\sigma_x)^2}{(\omega^2\mu_0\epsilon - j\omega\mu_0\sigma_\perp - k^2)}\right\}. \quad (14)$$

In general, k_T^2 is a complex quantity. However, if $\mu_e B_0 \gg 1$, σ_x will be almost a pure real quantity and $\sigma_\perp$ will be negligibly small. Consequently, the imaginary part of k_T^2 will be very small. Therefore, the dissipative term that is due to the imaginary part of k_T will be treated as a perturbation.

Equation (13) can now be solved by means of the method of separation of variables.[21],[22] In view of the boundary conditions where $H_z(0, \phi)$ is finite and $H_z(r, \phi+2\pi) = H_z(r, \phi)$, we get

$$H_z = IJ_n(k_T r)e^{\pm jn\phi} \quad (15)$$

where $n = 0, 1, 2, 3, \cdots$, I is constant, and $J_n(k_T r)$ is the nth order Bessel function of the first kind. The waves characterized by $e^{+jn\phi}$ and $e^{-jn\phi}$ represent the left- and right-hand circular polarized waves, respectively.

Substituting (15) into Maxwell's equations and making use of the fact that $\sigma_\parallel$ is very large and E_z is very small, although it is not identically equal to zero, we obtain

$$E_{r\pm nm} = \pm V_{\pm nm}(n/k_{Tnm}r)(k_{\pm nm}/k_{Tnm})J_n(k_{Tnm}r)e^{\pm jn\phi} \quad (16)$$

$$E_{\phi\pm nm} = jV_{\pm nm}(k_{\pm nm}/k_{Tnm})J_n'(k_{Tnm}r)e^{\pm jn\phi} \quad (17)$$

$$E_{z\pm nm} \approx 0 \quad (18)$$

$$H_{r\pm nm} = -jI_{\pm nm}(k_{\pm nm}/k_{Tnm})J_n'(k_{Tnm}r)e^{\pm jn\phi} \quad (19)$$

$$H_{\phi\pm nm} = \pm I(n/k_{Tnm}r)(k_{\pm nm}/k_{Tnm})J_n(k_{Tnm}r)e^{\pm jn\phi} \quad (20)$$

$$H_{z\pm nm} = I_{\pm nm}J_n(k_{Tnm}r)e^{\pm jn\phi} \quad (21)$$

where $V_{\pm nm} = (\omega\mu_0/k_{\pm nm})\, I_{\pm nm}$, and k_{Tnm} is determined by the boundary condition $E_\phi(r_0, \phi) = 0$. Since k_T has many solutions that satisfy $J_n'(k_T r_0) = 0$, the subscript nm is used to designate mth root of $J_n'(k_T r) = 0$. Note that the elementary wave functions obtained resemble TE modes in an empty circular guide although they are not of identical form. Thus, it was found that the right- and left-hand circularly polarized quasi-TE mode waves exist in a circular solid-state plasma waveguide under a longitudinal magnetic field, provided that $\mu_e B_0 \gg 1$. The left- and right-hand circularly polarized quasi-TE waves are nondegenerate, i.e., the propagation constants k_{+nm} and k_{-nm} have different values. We also note that the characteristic wave impedance of each mode is given by

$$Z_{\pm nm} = E_{r\pm nm}/H_{\phi\pm nm} = -E_{\phi\pm nm}/H_{r\pm nm} = \omega\mu_0/k_{\pm nm}. \quad (22)$$

III. Propagation Constant

In Section II, the field configurations in the solid-state plasma waveguide were obtained and it was found that two types of circularly polarized quasi-TE waves exist. In this section, the propagation constants of these waves are calculated by means of a variational method. It is shown in the Appendix that the propagation constant in the solid-state plasma waveguide can be approximated by

$$k^2 = \left[\omega^2\mu_0 \iint_S \epsilon \mathbf{e}_1^* \cdot \mathbf{e}_1 dS - \iint_S (\nabla_t \times \mathbf{e}_1^*)\cdot(\nabla_t \times \mathbf{e}_1)dS - j\omega\mu_0 \iint_S \mathbf{e}_1^* \cdot (\bar{\bar{\sigma}} \cdot \mathbf{e}_1)dS\right] \cdot \left[\iint_S (\mathbf{a}_z \times \mathbf{e}_1^*)\cdot(\mathbf{a}_z \times \mathbf{e}_1)dS\right]^{-1} \quad (23)$$

where $\mathbf{e}_1$ is the electric field inside the waveguide, $\mathbf{e}_1^*$ is the complex conjugate of $\mathbf{e}_1^*$, and the surface integrals are taken over the cross section of the waveguide.

We use the field configurations obtained in Section II as trial fields. Substituting the tensor conductivity given by (1) and the electric field given by (16) through (18) into (23), we get

$$k^2_{\pm nm} = \omega^2\mu_0\epsilon - k^2_{Tnm} - j\omega\mu_0\sigma_\perp \pm \omega\mu_0\sigma_x A_{nm} \quad (24)$$

$$A_{nm} = \left[2n\int_0^{x'_{nm}} J_n J_n' dx\right] \cdot \left[\int_0^{x'_{nm}} \{(n^2/x)J_n^2 + xJ_n'^2\} dx\right]^{-1} \quad (25)$$

where $k_{Tnm}r = x$ and $k_{Tnm}r_0 = x_{nm}'$. Evaluating the integrals, we obtain

$$A_{nm} = [2nJ_n^2][x_{nm}'^2 J_{n-1}^2 - 2(n-1)J_{n-1}J_n + (x_{nm}'^2 - 2n)J_n^2]^{-1} \quad (26)$$

where the Bessel functions J_n and J_{n-1} are evaluated at $x = x_{nm}' = k_{Tnm}'r_0$. Numerical values of A_{nm} are computed and tabulated in Table I.

It can be seen that the left- and right-hand circularly polarized quasi-TE waves in the solid-state plasma waveguide are indeed nondegenerate, i.e., $k_{+nm} \neq k_{-nm}$, except for $n = 0$. When $n = 0$, $A_{0m} = 0$. Physically, this means that there is circular symmetry when $n = 0$.

Fig. 3 shows typical variations of the propagation constants of the lowest two modes of the left- and right-hand circularly polarized waves in the solid-state plasma waveguide. The propagation constants are normalized to $k_{\pm nm}/k_0$ where $k_0 = \omega\sqrt{\mu_0\epsilon}$, and the magnetic field is normalized to $\mu_e B_0/(\sigma_0/\omega\epsilon)$ where $\sigma_0 = qn_e\mu_e$ is the dc conductivity. The case where $k_0 = 1.5\ k_{T11}$ is taken as an example. The dotted lines show the cutoff regions, and $\beta_{\pm nm}$ and $\alpha_{\pm nm}$ are, respectively, the real and imaginary parts of $k_{\pm nm}$. The difference between k_{+nm} and k_{-nm} decreases as the magnetic field increases. This is due to the fact that σ_x decreases as the magnetic field increases as shown in Fig. 1.

TABLE I
NUMERICAL VALUES OF A_{nm}

m \ n	0	1	2	3	4	5
1	0	0.837	0.753	0.703	0.654	0.411
2	0	0.070	0.098	0.108	0.102	0.100
3	0	0.028	0.042	0.049	0.052	0.054

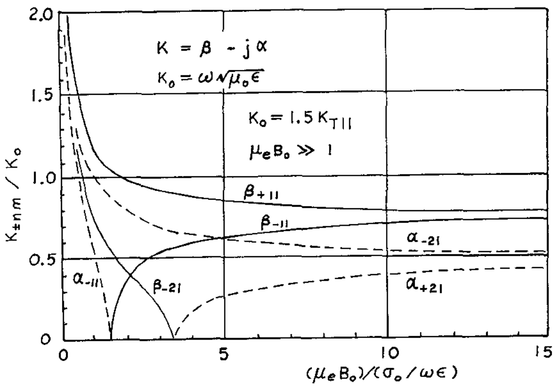

Fig. 3. Typical variations of the propagation constants of the quasi-$TE_{\pm 11}$ and $TE_{\pm 21}$ modes as a function of applied magnetic field in a circular solid-state plasma waveguide.

The off-diagonal term σ_x of the conductivity tensor is an odd function of B_0, thus it can be either positive or negative depending on the direction of the magnetic field. Therefore, a left-hand circularly polarized wave has a slower phase velocity than the right-hand circularly polarized waves of the same order, if the magnetic field is directed in the positive z direction regardless of the direction of propagation. If, however, the direction of the magnetic field is reversed, the phase velocity of the left-hand circularly polarized wave becomes faster than that of the right-hand circularly polarized wave.

IV. FARADAY EFFECT

Let us consider a circular solid-state plasma waveguide of uniform cross section as shown in Fig. 2. The regions $z<0$ and $z>\ell$ are empty and the region $0<z<\ell$ is filled with a solid-state plasma. The waveguide is placed under a longitudinal magnetic field B_0.

Suppose that a linearly polarized TE_{11} wave is excited in the first region $z<0$ of the waveguide. (Only the dominant modes are considered in this section.) The transverse components of the electric and magnetic fields of the linearly polarized TE_{11} wave in this region may be given by

$$\boldsymbol{E}_1 e^{-jk_1 z} = V_1[\boldsymbol{a}_r(1/k_{T11}r)J_1(k_{T11}r)\cos\phi + \boldsymbol{a}_\phi J_1'(k_{T11}r)\sin\phi]e^{-jk_1 z} \tag{27}$$

$$\boldsymbol{H}_1 e^{-jk_1 z} = I_1[\boldsymbol{a}_r(-1)J_1'(k_{T11}r)\sin\phi + \boldsymbol{a}_\phi(1/k_{T11}r)J_1(k_{T11}r)\cos\phi]e^{-jk_1 z} \tag{28}$$

where the subscript 1 refers to the first region $z<0$ of the waveguide. The propagation constant k_1 and the characteristic wave impedance Z_1 are, respectively, given by

$$k_1^2 = \omega^2\mu_0\epsilon_0 - k_{T11}^2 \tag{29}$$

$$Z_1 = V_1/I_1 = \omega\mu_0/k_1 \tag{30}$$

where $k_{T11}=1.8411/r_0$.

We can decompose $\boldsymbol{E}_1$ into the left- and right-hand circularly polarized TE_{11} waves, i.e.,

$$\boldsymbol{E}_1 = \boldsymbol{E}_{+1} + \boldsymbol{E}_{-1} \tag{31}$$

where

$$\boldsymbol{E}_{\pm 1} = (V_1/2)[\boldsymbol{a}_r(1/k_{T11}r)J_1(k_{T11}r) \pm j\boldsymbol{a}_\phi J_1'(k_{T11}r)]e^{\pm j\phi}.$$

The two circularly polarized waves given by (31) are degenerate in the empty sections of the waveguide.

In the region $0<z<\ell$ which is filled with a solid-state plasma, nondegenerate left- and right-hand circularly polarized quasi-TE waves exist as shown in the preceding sections. The propagation constants $k_\pm$ and the characteristic wave impedances $Z_\pm$ of the dominant modes of the two circularly polarized waves are, respectively, given by

$$k_\pm^2 = \omega^2\mu_0\epsilon - k_{T11}^2 - j\omega\mu_0\sigma_\perp \pm 0.837\omega\mu_0\sigma_x$$
$$Z_\pm = \omega\mu_0/k_\pm. \tag{32}$$

Since the wave impedances are discontinuous at $z=0$ and $z=\ell$, the waves experience multiple reflections at these discontinuities. By means of the impedance transformation method of the transmission line theory, it is found that the amount of the waves transmitted through the solid-state plasma into the third region $z>\ell$ of the waveguide will be

$$\boldsymbol{E}_{\pm 3} = T_\pm e^{j\psi_\pm}\boldsymbol{E}_{\pm 1} \tag{34}$$

$$T_\pm e^{j\psi_\pm} = \left[\frac{(1-\Gamma_{1\pm})(1+\Gamma_{2\pm})e^{-jk_\pm \ell}}{1-\Gamma_{1\pm}\Gamma_{2\pm}e^{-2k_\pm \ell}}\right] \tag{35}$$

where $\Gamma_{1\pm}$ and $\Gamma_{2\pm}$ are the reflection coefficients of the circularly polarized waves at the first and second waveguide discontinuities, i.e.,

$$\Gamma_{1\pm} = \frac{Z_1 - Z_\pm}{Z_1 + Z_\pm} = \frac{k_\pm - k_1}{k_\pm + k_1}$$

$$\Gamma_{2\pm} = \frac{Z_3 - Z_\pm}{Z_3 + Z_\pm} = \frac{k_\pm - k_3}{k_\pm + k_3}.$$

In general $k_\pm$, $\Gamma_{1\pm}$, and $\Gamma_{2\pm}$ are complex quantities, and $T_\pm$ and $\psi_\pm$, respectively, represent the magnitude and the phase shift of the transmission coefficients of the two circularly polarized waves through the solid-state plasma in the waveguide.

The total electric field of the transmitted wave thus becomes

$$\begin{aligned} E_3 &= E_{+3} + E_{-3} \\ &= (V_1/2)e^{j\psi}[(T_+ + T_-)\{a_r(1/k_{T11}r)J_1(k_{T11}r)\cos(\phi-\theta) \\ &\quad - a_\phi J_1'(k_{T11}r)\sin(\phi-\theta)\} \\ &\quad + j(T_+ - T_-)\{a_r(1/k_{T11}r)J_1(k_{T11}r)\sin(\phi-\theta) \\ &\quad + a_\phi J_1'(k_{T11}r)\cos(\phi-\theta)\}] \end{aligned} \tag{36}$$

where $\theta=(\psi_- - \psi_+)/2$ and $\psi=(\psi_-+\psi_+)/2$. The wave represented by (36) consists of two components. The planes of polarization of the two components of E_3 are oriented perpendicular to each other, and the two components are $\pi/2$ radians out of phase. In other words, the wave E_3 is elliptically polarized. It should also be noted that the direction of the major axis of polarization is rotated through an angle θ with respect to the original plane of polarization of the incident wave E_1 which is given by (27). This phenomenon is the microwave Faraday effect in the solid-state plasma waveguide. The angle of the Faraday rotation θ and the ellipticity of polarization E, which is defined as the ratio of the minor axis to the major axis of polarization, are thus given by

$$\theta = \frac{\psi_- - \psi_+}{2} \tag{37}$$

$$E = \frac{T_+ - T_-}{T_+ + T_-}. \tag{38}$$

The transmission coefficient and the insertion loss L of the solid-state plasma waveguide are

$$T = \frac{T_+ + T_-}{2} \tag{39}$$

$$L = -20 \log T. \tag{40}$$

It can be seen that θ and E are not necessarily equal to $(\beta_+ - \beta_-)(\ell/2)$ and $\tanh(\alpha_- - \alpha_+)(\ell/2)$, respectively, as given by (3) and (4), but they are strongly dependent on the reflection coefficients $\Gamma_{1\pm}$ and $\Gamma_{2\pm}$. Thus, the reflections from the waveguide discontinuities as well as the geometrical factors have significant influences on the Faraday rotation in the solid-state plasma waveguide.

V. Experiment

In order to conduct the experimental investigation of the microwave Faraday effect in a solid-state plasma waveguide, it is first necessary to select a solid-state plasma which has high electron mobility μ_e, but relatively high resistivity so that we can obtain a relatively large Hall angle, i.e., $\mu_e B_0 \gg 1$. It is also necessary to operate in a relatively high-frequency microwave region so that the displacement current will be larger than the conduction current, i.e., $\omega^2\mu_0\epsilon > \omega\mu_0\sigma_x \gg \omega\mu_0\sigma_\perp$.

An n-type indium antimonide single crystal was chosen since it has high electron mobility μ_e, a large relaxation time constant τ_e, and small electron effective mass m_e. At liquid nitrogen temperature, the InSb crystal used in the experiment has the following properties: $n_e = 10^{14}$ cm^{-3}, $\mu_e = 7\times10^5$ cm$^2\cdot$V$^{-1}\cdot$s^{-1}, $m_e = 0.013m_0$, $\tau_e = 5.2\times10^{-12}$ seconds, and $\epsilon_r = 17$. An experimental solid-state plasma waveguide as shown in Fig. 4 was constructed. The experimental waveguide consists of a circular waveguide in which a disk of the InSb crystal is mounted and coupled to rectangular waveguides at both ends by means of tapered transitions. The tapered transitions are used to transform the TE_{10} mode in the rectangular guide into the TE_{11} mode in the circular guide or vice versa. The orientations of the two rectangular guides can be rotated with respect to each other to locate the maximum and minimum transmissions.

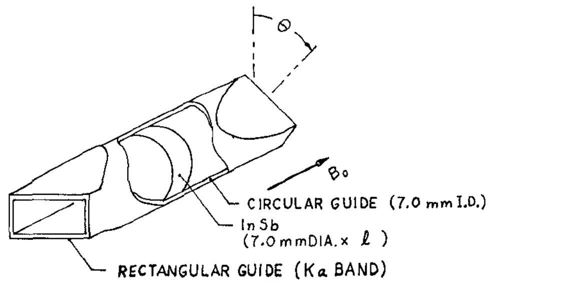

Fig. 4. An experimental solid-state plasma waveguide.

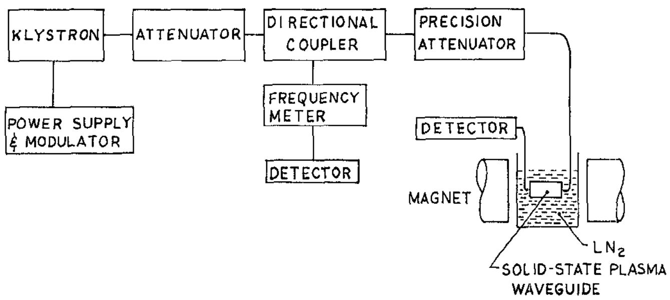

Fig. 5. Experimental setup.

The experimental setup shown in Fig. 5 was used to investigate the Faraday effect in the solid-stage plasma waveguide. The frequency was tuned to about 36 GHz. The experimental waveguide was placed between the poles of a magnet. At room temperature no transmission was detected even with the magnetic field applied. When the waveguide was cooled to 77°K with liquid nitrogen but with the magnetic field still zero, there was no transmission through the solid-state plasma waveguide. As the magnetic field was increased, at liquid nitrogen temperature, the transmission through the waveguide gradually increased. By rotating the relative orientations of the rectangular guides with respect to each other, positions and amplitudes of maximum and minimum transmission were measured under various magnetic fields. From these measurements, amount of the Faraday rotation, ellipticity of polarization of the transmitted wave, and the insertion loss of the solid-state plasma waveguide were obtained. The direction of the magnetic field was reversed, then the same measurements were repeated. The reverse magnetic field resulted in reversing the direction of rotation of polarization.

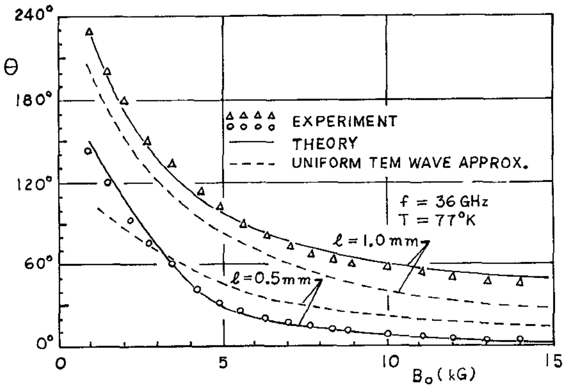

Fig. 6. Angle of Faraday rotation θ as a function of the applied magnetic field B_0 in a solid-state plasma waveguide.

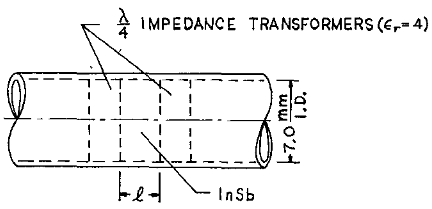

Fig. 9. An experimental solid-state plasma waveguide with quarter-wavelength impedance transformers.

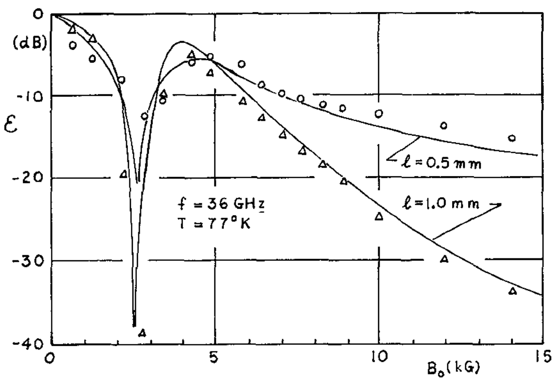

Fig. 7. Ellipticity of polarization E of the transmitted wave as a function of the applied magnetic field B_0 in a solid-state plasma waveguide.

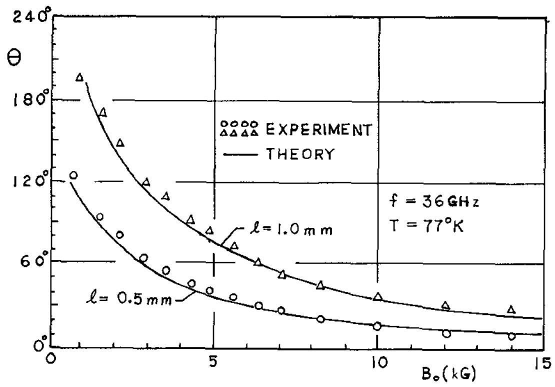

Fig. 10. Angle of Faraday rotation θ as a function of the applied magnetic field B_0 in a solid-state plasma waveguide with quarter-wavelength impedance transformers.

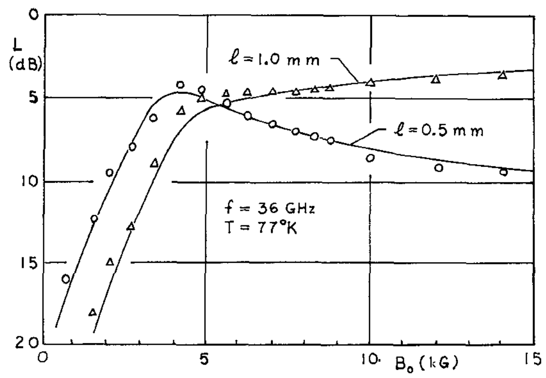

Fig. 8. Insertion loss L of a solid-state plasma waveguide as a function of the applied magnetic field B_0.

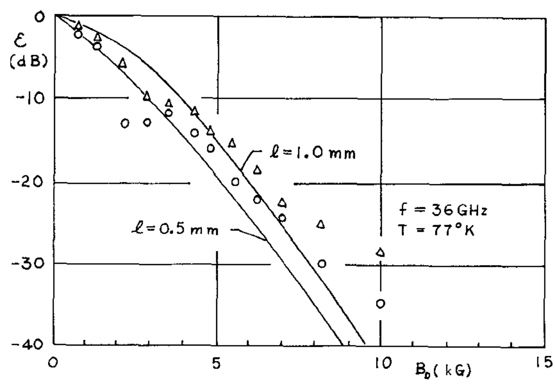

Fig. 11. Ellipticity of polarization E of the transmitted wave as a function of the applied magnetic field B_0 in a solid-state plasma waveguide with quarter-wavelength impedance transformers.

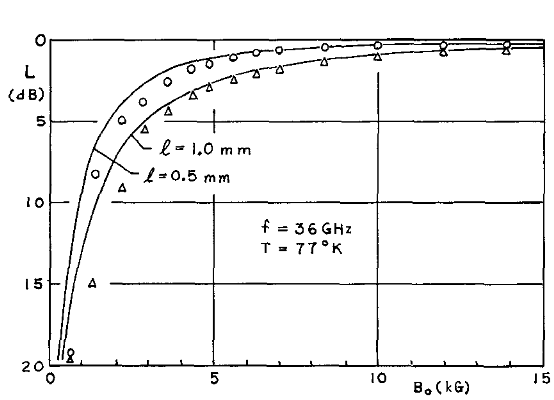

Fig. 12. Insertion loss L of a solid-state plasma waveguide with quarter-wavelength impedance transformers as a function of applied magnetic field B_0.

These sets of measurements were taken for two disks of the InSb crystal of different lengths, 0.5 and 1.0 mm. The results of the measurements are shown by small circles and small triangles in Figs. 6 through 8, together with the theoretical values (shown by solid lines) which were calculated from (37), (38), and (40). The simple TEM wave approximations which were obtained from (3) and (4) are shown by dotted lines for comparison purposes.

It was found in Section IV that reflections from the waveguide discontinuities have a significant influence on the Faraday effect in the solid-state plasma waveguide. By placing quarter-wavelength impedance transformers with $\epsilon_r=4$ as shown in Fig. 9, the reflection coefficients $\Gamma_{1\pm}$ and $\Gamma_{2\pm}$ at the surfaces of the InSb crystal disk were significantly reduced. The measurements of the Faraday rotation, ellipticity of polarization, and the insertion loss were repeated. The results of the measurements are shown with the theoretical curves in Figs. 10 through 12. Comparison of the experimental results obtained with the waveguide with the quarter-wavelength impedance transformers (Figs. 10 through 12) with the corresponding data obtained with the waveguide without the impedance transformers (Figs. 6 through 8) shows the significant influence of the wave reflections from the waveguide discontinuities on the microwave Faraday effect in the solid-state plasma waveguide.

VI. Conclusion

We have theoretically analyzed the characteristics of microwave propagation in a circular solid-state plasma waveguide under a longitudinal dc magnetic field. It was found that left- and right-hand circularly polarized quasi-TE waves exist and can propagate with very small attenuation in the solid-state plasma waveguide, provided that the magnetic field is relatively large, i.e., $\mu_e B_0 \gg 1$. By means of a variational method, the propagation constant of each quasi-TE wave was obtained. It was found that the two types of circularly polarized quasi-TE waves in the solid-state plasma waveguide are nondegenerate.

The analysis of the microwave Faraday effect in a solid-state plasma waveguide under a relatively high magnetic field presented in this paper shows that the angle of the Faraday rotation decreases as the magnetic field is increased, contrary to the Faraday effect in a small magnetic field region. This is due to the fact that the off-diagonal term of the tensor conductivity, viz., σ_x, decreases as the magnetic field is increased in the high magnetic field region. It was shown that the reflections from the waveguide discontinuities, viz., at the surfaces of the solid-state plasma, have a significant influence on the Faraday effect. It was also shown that the geometrical factors, viz., A and k_T in (30), must be taken into consideration in analyzing the Faraday effect in the waveguide.

The Faraday effect in a solid-state plasma waveguide was also experimentally investigated using n-type InSb single crystals. A large amount of Faraday rotation with very small attenuation of power through the solid-state plasma waveguide was obtained. The effect of wave reflections on the Faraday rotation in the waveguide was also experimentally observed. Comparison of the experimental results with the theory shows good agreement.

Microwave Faraday effect can also be found in magnetized ferrites[20],[23],[24] although the Faraday effect in ferrites is due to bound electrons, whereas in gaseous and solid-state plasmas it is due to free electrons and holes. It may be expected that, in a waveguide filled with a ferrite or a gaseous plasma, geometrical factors and reflections from the waveguide discontinuities will have similar influences on the Faraday effect as in a solid-state plasma waveguide.

The nonreciprocal property of the Faraday rotation in the solid-state plasma which is accompanied by small attenuation of power can be used effectively to develop a various nonreciprocal microwave device.[27],[28] Since the theoretical upper frequency of operation of a solid-state plasma when it is used as microwave devices appears at the cyclotron resonant frequency, which is higher than 1000 GHz in this case, the Faraday effect in a solid-state plasma may find important nonreciprocal microwave device applications in high-frequency regions such as millimeter and submillimeter waves.

Appendix

Variational Formula for the Propagation Constant

In order to derive an approximate expression for the propagation constant in a solid-state plasma waveguide, consider Maxwell's equations given by (6a) and (6b). Eliminating $\boldsymbol{H}$, we get

$$\nabla \times \nabla \times \boldsymbol{E} = \omega^2\mu_0\epsilon E - j\omega\mu_0\bar{\boldsymbol{\sigma}}\cdot\boldsymbol{E}. \tag{41}$$

Scalarly multiplying this by $\boldsymbol{E}^*$ (the complex conjugate of $\boldsymbol{E}$), we get

$$\boldsymbol{E}^*\cdot\nabla \times \nabla \times \boldsymbol{E} = \omega^2\mu_0\epsilon\boldsymbol{E}^*\cdot\boldsymbol{E} - j\omega\mu_0\boldsymbol{E}^*\cdot(\bar{\boldsymbol{\sigma}}\cdot\boldsymbol{E}.) \tag{42}$$

The wave propagating in the positive z direction can be expressed by

$$\boldsymbol{E}(r, \phi, z) = \boldsymbol{e}_1(r, \phi)e^{-jkz}. \tag{43}$$

Substituting this into (42) and integrating the result over the cross section S of the waveguide, and making use of vector identities and the divergence theorem, we obtain

$$\begin{aligned} k^2 &\iint_S (\boldsymbol{a}_z \times \boldsymbol{e}_1^*)\cdot(\boldsymbol{a}_z \times \boldsymbol{e}_1)dS \\ &+ \iint_S (\nabla_t \times \boldsymbol{e}_1^*)\cdot(\nabla_t \times \boldsymbol{e}_1)dS \\ &+ \oint_C \{(\nabla_t \times \boldsymbol{e}_1)\cdot(\boldsymbol{n} \times \boldsymbol{e}_1) \\ &- jk(\boldsymbol{a}_z \times \boldsymbol{e}_1)\cdot(\boldsymbol{n} \times \boldsymbol{e}_1^*)\}d\ell \\ &- jk\iint_S \{(\boldsymbol{a}_z \times \boldsymbol{e}_1)\cdot(\nabla_t \times \boldsymbol{e}_1^*) \\ &- (\nabla_t \times \boldsymbol{e}_1)\cdot(\boldsymbol{a}_z \times \boldsymbol{e}_1^*)\}dS \\ &= \omega^2\mu_0\iint_S \epsilon\boldsymbol{e}_1^*\cdot\boldsymbol{e}_1 dS - j\omega\mu_0\iint_S \boldsymbol{e}_1^*\cdot(\bar{\boldsymbol{\sigma}}\cdot\boldsymbol{e}_1)dS \end{aligned} \tag{44}$$

where C is the contour along the waveguide wall enclosing the cross section S, and n is the outgoing normal unit vector.

In Section II, it was found that there are quasi-TE modes in the solid-state plasma waveguide under a longitudinal magnetic field. In view of this fact and assumption 5) in Section I, we may choose the trial field $\mathbf{e}_1$, such that the z component of $\mathbf{e}_1$ is negligibly small everywhere in S and that the tangential component vanishes at the waveguide walls, i.e., $\mathbf{e}_1 \cdot \mathbf{a}_z \approx 0$ in S and $\mathbf{n} \times \mathbf{e}_1 = 0$ on C. Then, solving for k^2 in (44) we finally obtain

$$k^2 = \left[\omega^2 \mu_0 \iint_S \epsilon \mathbf{e}_1^* \cdot \mathbf{e}_1 dS - \iint_S (\nabla_t \times \mathbf{e}_1^*) \cdot (\nabla_t \times \mathbf{e}_1) dS - j\omega\mu_0 \iint_S \mathbf{e}_1^* \cdot (\bar{\bar{\sigma}} \cdot \mathbf{e}_1) dS \right] \cdot \left[\iint_S (\mathbf{a}_z \times \mathbf{e}_1^*) \cdot (\mathbf{a}_z \times \mathbf{e}_1) dS \right]^{-1} \cdot \quad (45)$$

This variational formula can be shown to be stationary provided that $\mathbf{n} \times \mathbf{e}_1 = 0$ on C.[28]

A similar expression for the propagation constant in a gaseous plasma was derived by Rao *et al.*[26] But the advantages of the expression given by (45) lie in the fact that it can be evaluated in terms of E field only.

References

[1] E. R. Wicher, "The influence of magnetic fields upon the propagation of electromagnetic waves in artificial dielectrics," *J. Appl. Phys.*, vol. 22, pp. 1327–1329, 1951.

[2] R. R. Rau and M. E. Caspari, "Faraday effect in germanium at room temperature," *Phys. Rev.*, vol. 100, pp. 632–639, 1955.

[3] J. K. Furdyna and S. Broersma, "Microwave Faraday effect in silicon and germanium," *Phys. Rev.*, vol. 120, pp. 1995–2003, 1960.

[4] H. E. M. Barlow, "Microwave Hall effect and the accompanying rotation of the plane of polarization," *Proc. IEE* (London), vol. 108, pp. 349–353, 1961.

[5] M. H. Engineer and B. R. Nag, "Influence of Hall effect on Faraday rotation," *J. Appl. Phys.*, vol. 36, pp. 3387–3388, 1965.

[6] ——, "Experimental observation of Faraday rotation in artificial dielectrics," *J. Appl. Phys.*, vol. 36, pp. 3388–3390, 1965.

[7] A. Bouwoknegt and J. Volger, "Measurement of the free carrier Faraday effect in n-type germanium at microwave frequencies," *Physica*, vol. 30, pp. 113–123, 1964.

[8] B. Donovan and J. Webster, "The Faraday effect in nondegenerate semiconductors," *Proc. Phys. Soc.*, vol. 78, pp. 120–132, 1961.

[9] B. Donovan and T. Medcalf, "The inclusion of multiple reflections in the theory of the Faraday effect in semiconductors," *Brit. J. Appl. Phys.*, vol. 15, pp. 1139–1151, 1964.

[10] A. Sommerfeld, *Optics*. New York: Academic Press, 1964, ch. 3.

[11] P. Aigrain, "Les 'Helicons' dans les semiconduteurs," *Proc. Internat'l Conf. on Semiconductor Physics, Prague, 1960*, Czechoslovakian Acad. of Sci. (Prague), pp. 224–226, 1961.

[12] A. Libchaber and R. Veilex, "Wave propagation in a gyromagnetic solid conductor: helicon waves," *Phys. Rev.*, vol. 127, pp. 774–776, 1962.

[13] S. J. Buchsbaum, "Theory of waves in solid state plasmas" in *Plasma Effects in Solids; Proceedings of the 7th International Conference on the Physics of Semiconductors, Paris, 1964.* New York: Academic Press, 1965, pp. 3–18.

[14] R. Bowers, "Propagation of electromagnetic waves in solids (experimental)," in *Plasma Effects in Solids; Proceedings of the 7th International Conference on the Physics of Semiconductors, Paris, 1964.* New York: Academic Press, 1965, pp. 19–35.

[15] J. P. Klozenberg, B. McNamara, and P. C. Thonemann, "The dispersion and attenuation of helicon waves in a uniform cylindrical plasma," *J. Fluid Mech.*, vol. 21, pp. 545–563, 1965.

[16] G. M. Harding and P. C. Thonemann, "A study of helicon waves in indium," *Proc. Phys. Soc.*, vol. 85, pp. 317–328, 1965.

[17] N. Perrin, B. Perrin, and W. Mercouroff, "Propagation of an electromagnetic wave in indium antimonide in the Q-band in a high dc magnetic field" in *Plasma Effect in Solids; Proceedings of the 7th International Conference on the Physics of Semiconductors, Paris, 1964.* New York: Academic Press, 1965, pp. 37–43.

[18] J. K. Furdyna, "Helicons, magnetoplasma edge, and Faraday rotation in solid-state plasmas at microwave frequencies," *Appl. Optics*, vol. 6, pp. 675–684, 1967.

[19] W. P. Allis, S. J. Buchsbaum, and A. Bers, *Waves in Anisotropic Plasmas.* Cambridge, Mass.: M.I.T. Press, 1962.

[20] A. G. Gurevich, *Ferrites at Microwave Frequencies.* Cambridge, Mass.: Boston Technical Publishers, 1965.

[21] R. E. Collin, *Field Theory of Guided Waves.* New York: McGraw-Hill, 1960.

[22] R. F. Harrington, *Time-Harmonic Electromagnetic Fields.* New York: McGraw-Hill, 1961.

[23] C. L. Hogan, "The ferromagnetic Faraday effect at microwave frequencies and its applications," *Rev. Mod. Phys.*, vol. 25, pp. 253–276, 1953.

[24] B. Lax and K. J. Button, *Microwave Ferrites and Ferrimagnetics.* New York: McGraw-Hill, 1962, ch. 12.

[25] L. Goldstein, "Nonreciprocal electromagnetic wave propagation in ionized gaseous media," *IRE Trans. Microwave Theory and Techniques*, vol. MTT-6, pp. 19–29, January 1958.

[26] K. V. Rao, J. T. Verdeyen, and L. Goldstein, "Interaction of microwaves in gaseous plasmas immersed in magnetic fields," *Proc. IRE*, vol. 49, pp. 1877–1889, December 1961.

[27] H. J. Kuno and W. D. Hershberger, "Solid-state plasma controlled nonreciprocal microwave device," *IEEE Trans. Microwave Theory and Techniques (Correspondence)*, vol. MTT-15, pp. 57–58, January 1967.

[28] H. J. Kuno, "Solid-state plasma waveguides: microwave propagation and Faraday effect," Ph.D. dissertation, Dept. of Engrg., University of California, Los Angeles, 1966.